DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION
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CONFORMAL GEOMETRY I

DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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Outline

PART I: OVERVIEW

PART II: SMOOTH THEORY

PART III: DISCRETIZATION

PART IV: ALGORITHMS
Part I: Overview

Discrete Conformal Geometry

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Motivation: Mapmaking Problem

• How do you make a flat map of the round globe?
• Hard to do! Like trying to flatten an orange peel...

Impossible without some kind of distortion and/or cutting.
Conformal Mapmaking

• Amazing fact: can always make a map that exactly preserves angles.

(Very useful for navigation!)
Conformal Mapmaking

• However, **areas** may be badly distorted…

(Greenland is not bigger than Australia!)
Conformal Geometry

More broadly, *conformal geometry* is the study of shape when one can measure only *angle* (not length).
Conformal Geometry—Visualized
Applications of Conformal Geometry Processing

Basic building block for many applications...

- Cartography
- Texture Mapping
- Remeshing
- Simulation
- 3D Fabrication
- Shape Analysis
- Sensor Networks
Why Conformal?

• Why so much interest in maps that preserve angle?
  • **QUALITY:** Every conformal map is already “really nice”
  • **SIMPlicity:** Makes “pen and paper” analysis easier
  • **Efficiency:** Often yields computationally easy problems
  • **Guarantees:** Well understood, lots of theorems/knowledge
Conformal Maps are “Really Nice”

- Angle preservation already provides a lot of regularity
- E.g., every conformal map has infinitely many derivatives ($C^\infty$)
- Scale distortion is smoothly distributed (harmonic)
Conformal Coordinates Make Life Easy

- Makes life easy “on pen and paper”
- **Curves:** life greatly simplified by assuming *arc-length* parameterization
- **Surfaces:** “arc-length” (isometric) not usually possible
  - conformal coordinates are “next best thing” (and always possible!)
  - only have to keep track of scale (rather than arbitrary Jacobian)
Aside: Isn’t Area-Preservation “Just as Good?”

• **Q:** What’s so special about angle? Why not preserve, say, area instead?

• **A:** Area-preservation alone can produce maps that are _nasty_!
  - Don’t even have to be smooth; huge space of possibilities.
  - E.g., any motion of an incompressible fluid (e.g., swirling water):

![Original](image1.png) ![Angle Preserving](image2.png) ![Area Preserving](image3.png)
Computing Conformal Maps is Efficient

- Algorithms boil down to efficient, scalable computation
  - sparse linear systems / sparse eigenvalue problems
  - convex optimization problems
- Compare to more elaborate mapping problems
  - bounded distortion, locally injective, etc.
  - entail more difficult problems (e.g., SOCP)
- Much broader domain of applicability
  - real time vs. “just once”
Conformal Maps Help Provide Guarantees

- Established topic*
  - lots of existing theorems, analysis
  - connects to standard problems (e.g., Laplace)
  - makes it easier to provide guarantees (max principle, Delaunay, etc.)
- **Uniformization theorem** provides (nearly) canonical maps

*Also makes it harder to do something truly new in conformal geometry processing…!
Discrete Conformal Maps?

To compute conformal maps, we need some finite “discretization.”

First attempt: preserve corner angles in a triangle mesh:
Rigidity of Angle Preservation

**Problem:** One triangle determines the entire map! (Too “rigid”)

Need a different way of thinking...
(Some) Characterizations of Conformal Maps

- angle preservation
- metric rescaling
- conjugate harmonic functions
- preservation of circles
- critical points of Dirichlet energy
(Some) Conformal Geometry Algorithms

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Some Key Ideas in Conformal Surface Geometry

Möbius Transformations / Stereographic Projection

Riemann Mapping / Uniformization

Cone Singularities

Cauchy-Riemann Equation

Ricci Flow / Cherrier Formula

Dirac Equation
PART II: SMOOTH THEORY

DISCRETE CONFORMAL GEOMETRY

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Conformal Maps of Surfaces

**Plane to Plane**

**Surface to Plane**

**Surface to Surface**
Why Not Higher Dimensions?

**Theorem** (Liouville). For $n \geq 3$, the only angle-preserving maps from $\mathbb{R}^n$ to itself (or from a region of $\mathbb{R}^n$ to $\mathbb{R}^n$) are Möbius transformations.

**Key idea:** conformal maps of volumes are *very* rigid.
Plane to Plane
Plane to Plane

• Most basic case: conformal maps from region of 2D plane to 2D plane.
• Basic topic of complex analysis
• Fundamental equation: Cauchy-Riemann
• Many ideas we will omit (e.g., power series / analytic point of view)
Differential of a Map

- Basic idea we’ll need to understand: differential of a map
- Describes how to “push forward” vectors under a differentiable map
- (In coordinates, differential is represented by the Jacobian)

\[
\frac{df(X)}{h} = \lim_{h \to 0} \frac{f(p + hX) - f(p)}{h}
\]

Intuition: “how do vectors get stretched out?”
A map is conformal if two operations are equivalent:

1. rotate, then push forward vector

2. push forward vector, then rotate

(How can we write this condition more explicitly?)
Complex Numbers

- Not much different from the usual Euclidean plane
- Additional operations make it easy to express scaling & rotation
- Extremely natural for conformal geometry
- Two basis directions: 1 and $i$
- Points expressed as $z = a + bi$
Complex Numbers

More importantly: obscures geometric meaning.
Symbol $i$ denotes *quarter-turn* in the *counter-clockwise* direction.
Complex Arithmetic—Visualized

rectangular coordinates

\[ z = a + bi \]

addition

\[ z_1 + z_2 \]

multiplication

\[ z_1 z_2 \]
Complex Product

- Usual definition:

- Complex product distributes over addition. Hence,

\[
\begin{align*}
  z_1 & := a + bi \\
  z_2 & := c + di \\
  z_1 z_2 & = (a + bi)(c + di) \\
           & = (a + bi)c + (a + bi)di \\
           & = ac + bci + adi + bdi^2 \\
           & = (ac - bd) + (ad + bc)i
\end{align*}
\]

Ok, terrific… but what does it mean geometrically?
**Rectangular vs. Polar Coordinates**

**Rectangular**

\[ z = a + bi \]

**Polar**

\[ z = r(\cos \theta + i \sin \theta) = re^{i\theta} \]

**Euler’s Identity**

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

(In practice: just convenient shorthand!)
Rotations with Complex Numbers

• How can we express rotation?

• Let $u$ be any unit complex number: $u = e^{i\theta}$

• Then for any point $z = re^{i\phi}$ we have

$$uz = (e^{i\theta})(re^{i\phi}) = re^{i(\theta+\phi)}$$

(same radius, new angle)
Scaling with Complex Numbers

• How can we express scaling?
• Let $s$ be any real complex number: $s = a + 0i$
• Then for any point $z = re^{i\phi}$ we have
  \[ sz = (a + 0i)(re^{i\phi}) = are^{i\phi} \]
  (same angle, new radius)
More generally, consider any two complex numbers:

\[ z_1 := r_1 e^{i\theta_1} \]
\[ z_2 := r_2 e^{i\theta_2} \]

We can express their product as

\[ z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)} \]

• New angle is \textit{sum} of angles
• New radius is \textit{product} of radii

(Now forget the algebra and remember the \textit{geometry}!)
Conformal Map, Revisited

- A map is conformal if two operations are equivalent:
  1. rotate, then push forward vector
  2. push forward vector, then rotate

(How can we write this condition more explicitly?)
Consider a map \( f : \mathbb{C} \to \mathbb{C} \)

Then \( f \) is conformal as long as

\[
df(zX) = zdf(X)
\]

for all tangent vectors \( X \) and all complex numbers \( z \).

I.e., if it doesn’t matter whether you rotate/scale before or after applying the map.

\((df \text{ is “complex linear”})\)
Holomorphic vs. Conformal

• Important linguistic distinction: a *conformal map* is a holomorphic map that is “nondegenerate”, i.e., the differential is never zero.
Several equivalent ways of writing Cauchy-Riemann equation:

\[ df(zX) = zdf(X) \]

\[ df(iX) = idf(X) \]

\[ \bar{\partial}f = 0 \]

All express the same geometric idea!
Aside: Real vs. Complex Linearity

What if we just ask for real linearity?

$$\forall c \in \mathbb{R}, \quad df(cX) = cdf(X)$$

No angle preservation.

In fact, maps can be arbitrarily “ugly”. Why?

Because any differentiable $f$ trivially satisfies this property!
Example—Möbius Transformations (2D)

**Definition.** In 2D, a Möbius transformation is an orientation-preserving map taking circles to circles or lines, and lines to lines or circles. Algebraically, any Möbius transformation can be expressed as a map of the form

\[ z \mapsto \frac{az + b}{cz + d} \]

for complex constants \( ad \neq bc \).
Möbius Transformations “Revealed”

(Douglas Arnold and Jonathan Rogness)

https://www.ima.umn.edu/~arnold/moebius/
Sphere Inversion (nD)

\[ x \mapsto \frac{x - c}{|x - c|^2} \]

(Note: Reverses orientation—anticonformal rather than conformal)
Surface to Plane
Surface to Plane

- Map curved surface to 2D plane (“conformal flattening”)
- Surface does not necessarily sit in 3D
- Slight generalization: target curvature is constant but nonzero (e.g., sphere)
- Many different equations: Cauchy-Riemann, Yamabe, …
How do we express this condition formally?
Tangent Plane

- **Tangent vectors** are those that “graze” the surface
- **Tangent plane** is all the tangent vectors at a given point

![Diagram](image-url)
Differential of a Map from Surface to Plane

• Consider a map taking each point of a surface to a point in the plane.

• Differential says how tangent vectors get “stretched out” under this map.

(Really no different from plane to plane…)}
Complex Structure

- **Complex structure** $J$ rotates vectors in each tangent plane by 90 degrees.
- Analogous to complex unit $i$.
- E.g., $J \circ J = -id$.
- For a surface in $\mathbb{R}^3$:
  
  $$ JX = N \times X $$

  (where $N$ is unit normal)

**Motivation:** will enable us to define conformal maps from surface to plane.
Holomorphic Maps from a Surface to the Plane

Plane to plane:
\[ df(\mathbb{i}X) = \mathbb{i}df(X) \]

Surface to plane:
\[ df(\mathcal{J}X) = \mathbb{i}df(X) \]
Example—Stereographic Projection

How? Don’t memorize some formula—*derive it yourself!*

E.g., What’s the equation for a sphere? What’s the equation for a ray?
**Riemann Mapping Theorem**

**Theorem** (Riemann). Any nonempty simply-connected open proper subset of $\mathbb{C}$ can be conformally mapped to the unit open disk $D^2 := \{z \in \mathbb{C} : |z| < 1\}$.

**Fact.** The only conformal maps from $D^2$ to $D^2$ are Möbius transformations of the form

$$z \mapsto e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

where $a \in D^2$ and $\theta \in S^1$ (three degrees of freedom: inversion center and rotation).
Riemannian Metric

- Can also understand conformal maps in terms of Riemannian metric
- Riemannian metric $g$ is simply inner product in each tangent space
- Allows us to measure length, angle, etc.
- E.g., Euclidean metric is just dot product:
  \[ g_{\mathbb{R}^n}(X, Y) := \sum_i X_i Y_i \]
- In general, length and angle recovered via
  \[ |X| := \sqrt{g(X, X)} \]
  \[ \angle(X, Y) := \arccos\left(\frac{g(X, Y)}{|X||Y|}\right) \]
Conformally Equivalent Metrics

- Two metrics are *conformally equivalent* if they are related by a positive *conformal scale factor* at each point \( p \):
  \[
  \tilde{g}_p = e^{2u(p)} g_p \quad u : M \to \mathbb{R}
  \]

- Why write scaling as \( e^{2u} \)? Initially mysterious, but…
  - ensures scaling is always *positive*
  - factor \( e^u \) gives length scaling
  - more natural way of talking about *area distortion* (e.g., doubling in scale “costs” just as much as halving)

Q: Does this transformation preserve *angles*?
Uniformization Theorem

- Roughly speaking, Riemannian metric on any surface is conformally equivalent to one with \textit{constant curvature} (flat, spherical, hyperbolic).

\[ \tilde{g} = e^{2u} g \]
Why is Uniformization Useful?

- Provides canonical domain for solving equations, comparing data, cross-parameterization, etc.

- Careful: still have a few degrees of freedom (e.g., Möbius transformations)
Surface to Surface
Surface to Surface

- Conformal deformations of surfaces embedded in space
- Both surfaces can have arbitrary curvature (not just sphere, disk, etc.)
- Opens door to much broader geometry processing applications
- Very recent theory & algorithms (~1996/2011)
- Key equation: *time-independent Dirac equation*

Won’t say too much today… see https://youtu.be/UQC_emOPVK8
Geometry in the Quaternions

• Just as complex numbers helped with 2D transformations, *quaternions* provide natural language for 3D transformations.

• Recent use of quaternions as alternative way of analyzing surfaces (Pedit, Pinkall, and others).

• Basic idea: points \((a, b, c)\) get replaced with *imaginary* quaternions \(ai + bj + ck\).

• Surface is likewise an imaginary map \(f\).
Stretch Rotations

• How do we express rotation using quaternions?

• Similar to complex case, can rotate a vector $x$ using a unit quaternion $q$:

$$\tilde{x} = \bar{q}xq$$

• If $q$ has non-unit magnitude, we get a rotation and scaling

• Should remind you of conformal map: *scaling & rotation* (but no shear)
Spin Equivalence

• From here, not hard to express conformal deformation of surfaces

• Two surfaces $f_0, f$ are *spin equivalent* if their tangent planes are related by a pure scaling and rotation at each point:

$$d\tilde{f}(X) = \bar{\psi} \, df(X) \psi$$

for all tangent vectors $X$ and some stretch rotation $\psi : M \to IH$
Dirac Equation

• From here, one can derive the fundamental equation for conformal surface deformations, a time-independent Dirac equation

\[
(D - \rho)\psi = 0
\]

quaternionic Dirac operator

\[
D\psi := -\frac{df \wedge d\psi}{|df|^2}
\]

change in curvature

stretch rotation

CRANE, PINKALL, SCHRÖDER, “Spin Transformations of Discrete Surfaces” (2011)
Spin vs. Conformal Equivalence

• Two surfaces that are spin equivalent are also conformally equivalent: tangent vectors just get *rotated* and *scaled*! (no shearing)

• Are conformally equivalent surfaces always spin equivalent?
  • **No** in general, e.g., tori that are not *regularly homotopic* (below)
  • **Yes** for topological spheres
Why Not Just Optimize Angles?

- Forget the mathematics—why not just optimize mesh to preserve angles?
- As discussed before, angle preservation is too rigid!
- E.g., convex surface uniquely determined by angles (up to rigid motion)
Next Up...
Next up… Discretization & Algorithms

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